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NORM INEQUALITIES FOR HOLOMORPHIC SEMIGROUPS

BY

T. J. RANSFORD*

Département de mathématiques et de statistique, Université Laval
Québec (QC), Canada G1K 7P4
e-mail: ransford@mat.ulaval.ca

ABSTRACT

Let $(a^z)_{\mathrm{Re}\, z>0}$ be a holomorphic semigroup in a Banach algebra. Provided that a certain integral along the line $\mathrm{Re}\, z = 1$ is finite, it is possible to estimate $\|a^z\|$ purely in terms of this integral and the spectral radius of a. This generalizes earlier results of Esterle and Sinclair (used to prove tauberian theorems), which were applicable just to radical Banach algebras.

1. Introduction and main results

Throughout this paper, H denotes the open right half-plane:

$$H = \{z \in \mathbf{C}\colon \mathrm{Re}\, z > 0\}.$$

Let $(A, \|\cdot\|)$ be a Banach algebra, not necessarily having an identity. A family $(a^z)_{z\in H}$ of elements of A is called a **holomorphic semigroup** if the map $z \mapsto a^z\colon H \to A$ is holomorphic and satisfies

$$a^{z+w} = a^z a^w \qquad (z, w \in H). \tag{1}$$

We write a^1 simply as a, so that a^n has its usual meaning when n is a positive integer.

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Suppose now that A is a **radical** Banach algebra, i.e., that the spectral radius of every element of A is zero. Esterle [E, Theorem 2.4] showed that if $(a^z)_{z\in H}$ is a holomorphic semigroup in A satisfying

$$\log^+ \|a^z\| = O(|z|^\alpha) \quad \text{as } |z|\to\infty, \quad \operatorname{Re} z \geq 1, \tag{2}$$

where α is a constant with $\alpha < 1$, then in fact $a^z \equiv 0$. Subsequently, Sinclair [S, Theorem 5.6] improved this result by showing that the same conclusion follows on replacing (2) by the weaker hypothesis that

$$\int_{-\infty}^{\infty} \frac{\log^+ \|a^{1+iy}\|}{1+y^2}\, dy < \infty. \tag{3}$$

Apart from their intrinsic interest, these results have a beautiful application, namely a new proof of the Wiener tauberian theorem (see [E, §3] or [S, §5.5] for more details).

The purpose of this paper is to investigate the consequences of (3) when A is a general Banach algebra, no longer assumed radical. Our starting point is the following simple but very general inequality.

THEOREM 1.1: *Let $(A, \|\cdot\|)$ be a Banach algebra, and let $(a^z)_{z\in H}$ be a holomorphic semigroup in A satisfying* (3). *Then we have*

$$\|a^{z+2}\| \leq \rho\sigma\|a^z\| \qquad (z \in H), \tag{4}$$

where

$$\rho = \text{spectral radius of } a,$$
$$\sigma = \exp\left(\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log^+ \|a^{1+iy}\|}{1+y^2}\, dy\right).$$

Remarks: (i) If A is a radical Banach algebra, then necessarily $\rho = 0$, and so (4) implies that $a^z = 0$ whenever $\operatorname{Re} z > 2$. By the identity principle for holomorphic functions, it follows that $a^z \equiv 0$ on H. Thus Theorem 1.1 contains the results of Esterle and Sinclair as special cases. In fact our proof of Theorem 1.1 is also more elementary, because, unlike Esterle and Sinclair, we make no appeal to the Ahlfors–Heins theorem.

(ii) When A is no longer radical, Theorem 1.1 becomes a quantitative version of Sinclair's result. To appreciate the content of (4), it is helpful to compare the relative sizes of ρ, σ and $\|a^2\|$. There is a simple relation between these, namely

$$\rho^2 \leq \|a^2\| \leq \sigma^2. \tag{5}$$

The first inequality follows immediately from the fact that ρ^2 is the spectral radius of a^2, while the second is proved by integrating the inequality

$$\log \|a^2\| \leq \log^+ \|a^{1+iy}\| + \log^+ \|a^{1-iy}\|$$

with respect to the probability measure $dy/\pi(1+y^2)$ on $\mathbf{R}$, and then taking exponentials of both sides. Thus, for example, combining (1) and (5) we immediately see that

$$\|a^{z+2}\| \leq \sigma^2 \|a^z\| \qquad (z \in H).$$

The significance of Theorem 1.1 is that it replaces one of the σ's in this elementary inequality with a ρ, which, as we have already seen, is crucial for the application to the Esterle and Sinclair theorems.

The inequalities (4) and (5) have the interesting consequence that they yield bounds for even powers of a which depend only on ρ and σ, and not on the individual semigroup. Indeed, starting with a^{2n}, after $n-1$ applications of (4) we have $\|a^{2n}\| \leq (\rho\sigma)^{n-1}\|a^2\|$, and combining this with (5), we obtain the following corollary.

COROLLARY 1.2: *Under the assumptions of Theorem 1.1, we have*

$$\|a^{2n}\| \leq \rho^{n-1}\sigma^{n+1} \qquad (n \geq 1). \tag{6}$$

In fact this is a special case of the following more general result.

THEOREM 1.3: *Let $(A, \|\cdot\|)$ be a Banach algebra, and let $(a^z)_{z\in H}$ be a holomorphic semigroup in A satisfying (3). Then, writing $z = x + iy$, we have*

$$\|a^z\| \leq \rho^{\frac{1}{2}x-1}\sigma^{\frac{1}{2}x+\frac{\pi}{2}|y|+K} \qquad (\operatorname{Re} z \geq 2), \tag{7}$$

where ρ, σ are defined as in Theorem 1.1, and

$$K = 1 + \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2} \approx 2.166.$$

Further, if $\operatorname{Re} z$ is also an even integer, then (7) holds with $K = 1$.

The inequality (7) yields positive information even if ρ is replaced by σ. To put this in context, we recall that a key step in Sinclair's improvement of Esterle's original result was a lemma [S, Lemma 5.7], suggested by A. M. Davie, showing that the condition (3) automatically implies that the semigroup is of exponential type on the half-plane $\operatorname{Re} z \geq 3$. From (7) we can read off an improvement of this lemma, not apparently obtainable by the methods of [S].

COROLLARY 1.4: *Under the assumptions of Theorem* 1.3, *the function* $z \mapsto \|a^z\|$ *is of exponential type on the half-plane* $\operatorname{Re} z \geq 2$.

We do not know whether the 2 can be reduced any further, since the proof of (7) is valid only for $\operatorname{Re} z \geq 2$. However, we can prove an inequality valid for all $\operatorname{Re} z > 1$ at the cost of imposing an extra condition on the semigroup, namely that

$$\|a^z\| \not\to \infty \qquad \text{as } z \to 0. \tag{8}$$

This condition is automatically satisfied, for example, if A is the algebra of bounded operators on a complex Banach space and a^x tends strongly to the identity as $x \to 0^+$, as is the case for a C_0-semigroup (see [HP, §10.6]).

THEOREM 1.5: *Let* $(A, \|\cdot\|)$ *be a Banach algebra, and let* $(a^z)_{z\in H}$ *be a holomorphic semigroup in* A *satisfying* (3) *and* (8). *Then, writing* $z = x + iy$, *we have*

$$\|a^z\| \leq \rho^{x-1}\sigma^{(|z-2|+|z|)^2/4(x-1)} \qquad (\operatorname{Re} z > 1), \tag{9}$$

where ρ, σ *are defined as in Theorem* 1.1.

Which of the inequalities (7) and (9) is better? When x is large compared with $|y|$, the answer depends on the value of $\rho\sigma$: if $\rho\sigma > 1$ then the bound in (7) is better, while if $\rho\sigma < 1$ then (9) gives the sharper estimate. On the other hand, when $|y|$ is large compared with x, then (7) is always the better inequality, because the exponent of σ is linear in $|y|$, whereas in (9) it is quadratic.

Finally, we remark that putting $z = 2$ in (9) gives $\|a^2\| \leq \rho\sigma$, which leads to the following improvement of (6).

COROLLARY 1.6: *Under the assumptions of Theorem* 1.5, *we have*

$$\|a^{2n}\| \leq \rho^n\sigma^n \qquad (n \geq 1). \tag{10}$$

The strategy for proving all these theorems is the same: first establish a crude estimate using just the semigroup law (1) and the hypothesis (3), and then refine it by the methods of complex analysis, exploiting the fact that the semigroup is holomorphic. The crude estimates are derived in Section 2, the necessary tools from function theory are developed in Section 3, and then the proofs are completed in Section 4. Finally, in Section 5, we make some concluding remarks and pose a few questions.

2. Elementary estimates

In this section we derive some simple estimates which do not depend on holomorphicity. To emphasize this, the results will be stated for a **continuous semigroup**, namely a family $(a^z)_{z\in H}$ in A, still satisfying (1), for which the map $z \mapsto a^z$ is now merely continuous on H rather than holomorphic.

LEMMA 2.1: *Let $(A, \|\cdot\|)$ be a Banach algebra, and let $(a^z)_{z\in H}$ be a continuous semigroup in A. Then, given $w \in H$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $M > \inf_{r>0} \|a^{re^{i\theta}}\|^{1/r}$, there exists a constant C such that*

$$\|a^{w+re^{i\theta}}\| \leq CM^r \qquad (r \geq 0).$$

Proof: By definition of M, there exists $r_0 > 0$ such that

$$\|a^{r_0 e^{i\theta}}\| < M^{r_0}.$$

Given $r \geq 0$, write $r = kr_0 + s$, where k is a non-negative integer and $0 \leq s < r_0$. Then we have

$$\|a^{w+re^{i\theta}}\| \leq \|a^{w+se^{i\theta}}\| \|a^{r_0 e^{i\theta}}\|^k \leq \|a^{w+se^{i\theta}}\| M^{kr_0} \leq CM^r,$$

where $C = \sup_{0\leq s\leq r_0} \left(\|a^{w+se^{i\theta}}\|/M^s\right)$. ∎

Note that, by the spectral radius formula, we can choose M as close as we please to the spectral radius of $a^{e^{i\theta}}$. In particular, taking $\theta = 0$, we deduce the following corollary.

COROLLARY 2.2: *Under the assumptions of Lemma 2.1,*

$$\limsup_{r\to\infty} \|a^{w+r}\|^{1/r} \leq \rho,$$

where ρ is the spectral radius of a.

Lemma 2.1 can easily be adapted to yield bounds for $\sup_{|\theta|\leq\alpha} \|a^{w+re^{i\theta}}\|$, provided that $\alpha < \pi/2$ (see e.g. [S, §5.2]). On the other hand, to obtain estimates on a whole half-plane, we need to invoke an assumption on the growth of the semigroup. Here is a result of this kind, in which we assume that (3) holds. The resulting inequality, though weak, holds on a relatively large domain.

LEMMA 2.3: *Let $(A, \|\cdot\|)$ be a Banach algebra, and let $(a^z)_{z\in H}$ be a continuous semigroup in A satisfying* (3). *Then, given $w \in H$, there exists a constant C such that*

$$\|a^{w+1+z}\| \le Ce^{|z|^2} \qquad (z \in H).$$

Proof: Set

$$E = \{t \in \mathbf{R}\colon \log^+ \|a^{1+it}\| > (1+t^2)/2\} \qquad \text{and} \qquad I = \int_{-\infty}^{\infty} \frac{\log^+ \|a^{1+it}\|}{1+t^2}\,dt.$$

Then E is an open subset of $\mathbf{R}$, and has Lebesgue measure at most $2I$ because

$$I \ge \int_E \frac{\log^+ \|a^{1+it}\|}{1+t^2}\,dt \ge \int_E \frac{1}{2}\,dt.$$

Therefore, given $y \in \mathbf{R}$, there exists $t \in \mathbf{R} \setminus E$ with $|y-t| \le I$, and so

$$\begin{aligned}\|a^{\frac12 w+1+iy}\| &\le \|a^{1+it}\|\|a^{\frac12 w+i(y-t)}\| \\ &\le e^{\frac12+\frac12 t^2}\|a^{\frac12 w+i(y-t)}\| \\ &\le e^{\frac12+I^2+y^2}\|a^{\frac12 w+i(y-t)}\| \\ &\le C_1 e^{y^2},\end{aligned}$$

where $C_1 = e^{\frac12+I^2}\sup_{|s|\le I}\|a^{\frac12 w+is}\|$.

Also, applying Lemma 2.1 with $\theta = 0$, there exist constants C_2, M such that for all $x \ge 0$,

$$\|a^{\frac12 w+x}\| \le C_2 M^x = C_2 e^{x\log M} \le C_2 e^{x^2+(\log M)^2}.$$

Combining these inequalities, we deduce that for $z = x+iy \in H$,

$$\|a^{w+1+z}\| \le \|a^{\frac12 w+x}\|\|a^{\frac12 w+1+iy}\| \le C_2 e^{x^2+(\log M)^2} C_1 e^{y^2} = Ce^{|z|^2},$$

where $C = C_1 C_2 e^{(\log M)^2}$. ∎

Finally, in proving Theorem 1.3, we shall need the following extension of (5).

LEMMA 2.4: *Let $(A, \|\cdot\|)$ be a Banach algebra, and let $(a^z)_{z\in H}$ be a continuous semigroup in A satisfying* (3). *Then*

$$\|a^{2+iy}\| \le \sigma^{2+\frac{\pi}{2}|y|} \qquad (y \in \mathbf{R}),$$

where σ is defined as in Theorem 1.1.

Proof: We can suppose, without loss of generality, that $y > 0$. Then we have

$$\begin{aligned}\pi \log \sigma &= \int_{-\infty}^{y/2} \frac{\log^+ \|a^{1+it}\|}{1+t^2}\,dt + \int_{y/2}^{\infty} \frac{\log^+ \|a^{1+it}\|}{1+t^2}\,dt \\ &= \int_{y/2}^{\infty} \frac{\log^+ \|a^{1+i(y-t)}\|}{1+(y-t)^2}\,dt + \int_{y/2}^{\infty} \frac{\log^+ \|a^{1+it}\|}{1+t^2}\,dt \\ &\geq \int_{y/2}^{\infty} \frac{\log^+ \|a^{1+i(y-t)}\| + \log^+ \|a^{1+it}\|}{1+t^2}\,dt \\ &\geq \int_{y/2}^{\infty} \frac{\log^+ \|a^{2+iy}\|}{1+t^2}\,dt \\ &= \log^+ \|a^{2+iy}\| \arctan(2/y),\end{aligned}$$

and therefore

$$\|a^{2+iy}\| \leq \sigma^{\pi/\arctan(2/y)}.$$

To complete the proof, it thus suffices to show that

$$\frac{\pi}{\arctan(2/y)} \leq 2 + \frac{\pi}{2}y \qquad (y > 0).$$

This is equivalent to

$$\frac{1}{\arctan u} - \frac{1}{u} \leq \frac{2}{\pi} \qquad (u > 0),$$

which, in turn, is equivalent to

$$\frac{1}{\theta} - \frac{1}{\tan\theta} \leq \frac{2}{\pi} \qquad (0 < \theta < \pi/2).$$

However, since

$$\frac{d}{d\theta}\left(\frac{1}{\theta} - \frac{1}{\tan\theta}\right) = -\frac{1}{\theta^2} + \frac{1}{\sin^2\theta} > 0 \qquad (0 < \theta < \pi/2),$$

it follows that

$$\sup_{0<\theta<\pi/2}\left(\frac{1}{\theta} - \frac{1}{\tan\theta}\right) = \lim_{\theta\to\pi/2^-}\left(\frac{1}{\theta} - \frac{1}{\tan\theta}\right) = \frac{2}{\pi},$$

and so the desired inequality is indeed true. ∎

3. Some results from function theory

In this section we gather together the results that we shall need from complex analysis, starting with a simple lemma about harmonic functions.

LEMMA 3.1: *Let $\varphi\colon \partial H \to [0,\infty)$ be a continuous function such that*

$$\int_{-\infty}^{\infty} \frac{\varphi(it)}{1+t^2}\,dt < \infty,$$

and define $h\colon H \to [0,\infty]$ by

$$h(x+iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x}{x^2+(y-t)^2}\varphi(it)\,dt.$$

Then:

(i) *h is a positive harmonic function on H;*
(ii) *$\lim_{z\to\zeta} h(z) = \varphi(\zeta)$ for all $\zeta \in \partial H$;*
(iii) *$h(z) \le h(1)\big(|z-1|+|z+1|\big)^2/(4\operatorname{Re} z)$ for all $z \in H$.*

Proof: Both (i) and (ii) are standard (see e.g. [G, Chapter I, Lemma 3.3]). Part (iii) is a thinly disguised form of Harnack's inequality. Indeed, if we define $\widetilde{h}(w) = h\big((1+w)/(1-w)\big)$ for $|w|<1$, then $\widetilde{h}$ is a positive harmonic function on the unit disc, and so by the standard form of Harnack's inequality (see e.g. [R, Theorem 1.3.1]),

$$\widetilde{h}(w) \le \widetilde{h}(0)\frac{1+|w|}{1-|w|} \qquad (|w|<1).$$

Re-writing this in terms of $h(z)$ gives the inequality in (iii). ∎

We shall also need a version of the Phragmén–Lindelöf principle. In its classical form, this reads as follows (see e.g. [R, Theorem 2.3.7]).

PHRAGMÉN–LINDELÖF PRINCIPLE FOR SECTORS: *Let S be the sector*

$$S = \{z \in \mathbf{C}\setminus\{0\}\colon |\arg(z)-\theta| < \beta/2\},$$

where $\theta \in \mathbf{R}$ and $\beta \in (0,2\pi)$, and let $u\colon S \to [-\infty,\infty)$ be a subharmonic function satisfying:

(i) *$\limsup_{z\to\zeta} u(z) \le 0$ for all $\zeta \in \partial S$, and*
(ii) *$u^+(z) = O(|z|^\alpha)$ as $|z|\to\infty$, where $\alpha\beta < \pi$.*

Then $u \leq 0$ *on* S.

In particular, if S is the half-plane H, then $\beta = \pi$, and so we need $\alpha < 1$ for this theorem to be applicable. However, in our context, all that we shall know *a priori* is that $u^+(z) = O(|z|^2)$ on H. We therefore prove the following version of the result, in which the restriction on α is relaxed, albeit at the cost of an extra hypothesis on u.

LEMMA 3.2: *Let* $u: H \to [-\infty, \infty)$ *be a subharmonic function satisfying:*

(i) $\limsup_{z \to \zeta} u(z) \leq 0$ *for all* $\zeta \in \partial H$,

(ii) $u^+(z) = O(|z|^\alpha)$ *as* $|z| \to \infty$, *where* $\alpha < \infty$, *and*

(iii) $\limsup_{r \to \infty} u(re^{i\theta})/r < \infty$ *for each* $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Then we have

$$u(z) \leq L \operatorname{Re} z \qquad (z \in H),$$

where $L = \limsup_{r \to \infty} u(r)/r$.

Proof: For convenience, we extend u to $\overline{H}$ by setting $u \equiv 0$ on ∂H. The hypothesis (i) ensures that u, thus extended, is upper semicontinuous on $\overline{H}$.

Our first objective is to show that

$$u^+(z) = O(|z|) \qquad \text{as } |z| \to \infty. \tag{11}$$

Fix $\beta \in (0, 2\pi)$ with $\alpha\beta < \pi$. Take θ with $|\theta| \leq \pi/2 - \beta/2$, and let S_θ be the sector

$$S_\theta = \{z \in \mathbf{C} \setminus \{0\}: |\arg(z) - \theta| < \beta/2\}.$$

Then, using the hypothesis (iii) with θ replaced by $\theta \pm \beta/2$ (or, if one of these is $\pm\pi/2$, the fact that $u = 0$ on ∂H), there exist constants C_θ, D_θ such that

$$u(\zeta) \leq C_\theta + D_\theta \operatorname{Re} \zeta \qquad (\zeta \in \partial S_\theta).$$

Applying the Phragmén–Lindelöf principle to $u(z) - C_\theta - D_\theta \operatorname{Re} z$ on the sector S_θ, we deduce that

$$u(z) \leq C_\theta + D_\theta \operatorname{Re} z \qquad (z \in S_\theta).$$

As H can be covered by a finite number of such sectors S_θ, we conclude that (11) holds.

This now gives us the right to repeat the above argument with β replaced by any number less than π. In particular, the Phragmén–Lindelöf principle is applicable to u on the quadrants

$$H^+ = \{z \in H\colon \operatorname{Im} z > 0\} \qquad \text{and} \qquad H^- = \{z \in H\colon \operatorname{Im} z < 0\}.$$

Given $L > \limsup_{r\to\infty} u(r)/r$, there exists a constant $C > 0$ such that $u(r) \le C + Lr \ \ (r > 0)$. Together with the fact that $u \le 0$ on ∂H, this implies that

$$u(\zeta) \le C + L \operatorname{Re} \zeta \qquad (\zeta \in \partial H^+ \cup \partial H^-).$$

Applying the Phragmén–Lindelöf principle to $u(z) - C - L \operatorname{Re} z$ on each of H^+ and H^-, we deduce that

$$u(z) \le C + L \operatorname{Re} z \qquad (z \in H).$$

It follows that the function $u(z) - L \operatorname{Re} z$ is bounded above on H. This means that we can apply Phragmén–Lindelöf to it directly on the whole half-plane. Since $u(\zeta) - L \operatorname{Re} \zeta \le 0$ on ∂H, our conclusion is that

$$u(z) \le L \operatorname{Re} z \qquad (z \in H).$$

Finally, as this holds for each $L > \limsup_{r\to\infty} u(r)/r$, it is also true with $L = \limsup_{r\to\infty} u(r)/r$. ∎

Lastly, in the proof of Theorem 1.3 we shall also need a Phragmén–Lindelöf principle for strips. The standard result of this kind is the following (see e.g. [R, Theorem 2.3.5]).

Phragmén–Lindelöf Principle for Strips: *Let T be the strip*

$$T = \{z \in \mathbf{C}\colon c - 1 < \operatorname{Re} z < c + 1\},$$

where $c \in \mathbf{R}$, and let $u\colon T \to [-\infty, \infty)$ be a subharmonic function satisfying:

(i) $\limsup_{z\to\zeta} u(z) \le 0$ *for all* $\zeta \in \partial T$, *and*

(ii) $u^+(z) = O(e^{\alpha|z|})$ *as* $|z| \to \infty$, *where* $\alpha < \pi/2$.

Then $u \le 0$ on T.

Again, this will not be quite enough for our purposes. This time it is the hypothesis (i) which needs to be extended.

LEMMA 3.3: *Let T be the strip*

$$T = \{z \in \mathbf{C}: c-1 < \operatorname{Re} z < c+1\},$$

where $c \in \mathbf{R}$, and let $u: T \to [-\infty, \infty)$ be a subharmonic function satisfying:

(i) $\limsup_{z\to\zeta} u(z) \le |\operatorname{Im}\zeta|$ *for all* $\zeta \in \partial T$, *and*

(ii) $u^+(z) = O(e^{\alpha|z|})$ *as* $|z| \to \infty$, *where* $\alpha < \pi/2$.

Then we have

$$u(z) \le |\operatorname{Im} z| + 8\kappa/\pi^2 \qquad (z \in T),$$

where $\kappa = \sum_{k=0}^{\infty}(-1)^k(2k+1)^{-2}$ *(Catalan's constant).*

Remark: The proof will make it clear that the constant is sharp.

Proof: We can suppose, without loss of generality, that $c = 0$, so that $T = \{z: |\operatorname{Re} z| < 1\}$. Let $\omega_T(z,\cdot)$ denote the harmonic measure for T at a point $z \in T$, and define $h: T \to [0, \infty]$ by

$$h(z) = \int_{\partial T} |\operatorname{Im}\zeta|\, d\omega_T(z,\zeta) \qquad (z \in T).$$

Then, in particular, we have

$$h(0) = \frac{1}{4}\int_{-\infty}^{\infty} \frac{|\operatorname{Im}(-1+it)|}{\cosh(\pi t/2)}\, dt + \frac{1}{4}\int_{-\infty}^{\infty} \frac{|\operatorname{Im}(1+it)|}{\cosh(\pi t/2)}\, dt < \infty, \tag{12}$$

which implies h is a positive harmonic function on T, satisfying $\lim_{z\to\zeta} h(z) = |\operatorname{Im}\zeta|$ for all $\zeta \in \partial T$. (In essence, this is just a special case of Lemma 3.1, transformed under a conformal mapping.)

Now set $\widetilde{u} = u - h$. Then $\widetilde{u}$ is a subharmonic function on T, satisfying $\limsup_{z\to\zeta} \widetilde{u}(z) \le 0$ for all $\zeta \in \partial T$, and $\widetilde{u}^+(z) = O(e^{\alpha|z|})$ as $|z| \to \infty$, for some $\alpha < \pi/2$. Applying the Phragmén–Lindelöf principle for strips, we deduce that $\widetilde{u} \le 0$ on T, or in other words, that

$$u(z) \le h(z) \qquad (z \in T). \tag{13}$$

Thus, we need to estimate $h(z)$ for $z \in T$. Given $x + iy \in T$, we have

$$\begin{aligned} h(x+iy) &= \int_{\partial T} |\operatorname{Im}\zeta|\, d\omega_T(x+iy,\zeta) \\ &= \int_{\partial T} |\operatorname{Im}(\zeta+iy)|\, d\omega_T(x,\zeta) \\ &\le \int_{\partial T} (|\operatorname{Im}\zeta| + |y|)\, d\omega_T(x,\zeta), \end{aligned}$$

and so

$$h(x+iy) \le h(x) + |y| \qquad (x+iy \in T). \tag{14}$$

This leaves us with the problem of estimating $h(x)$ for $x \in T \cap \mathbf{R}$. Given $x+iy \in T$, a calculation like the one above shows that

$$\begin{aligned} h(x+iy) + h(x-iy) &= \int_{\partial T} \big(|\operatorname{Im}(\zeta+iy)| + |\operatorname{Im}(\zeta-iy)|\big)\, d\omega_T(x,\zeta) \\ &\ge \int_{\partial T} |\operatorname{Im}(2\zeta)|\, d\omega_T(x,\zeta) = 2h(x). \end{aligned}$$

Hence, if we set $v(z) = -h(z) - h(\overline{z})$, then v is a subharmonic function on T satisfying

$$\sup_{y\in\mathbf{R}} v(x+iy) = -2h(x) \qquad (x \in T \cap \mathbf{R}).$$

Applying the Phragmén–Lindelöf principle to v on the strip $\{w\colon |\operatorname{Re} w| < x\}$, where $0 < x < 1$, we deduce that

$$v(w) \le \max\big(-2h(x), -2h(-x)\big) \qquad (|\operatorname{Re} w| < x).$$

Since $v(0) = -2h(0)$ and h is an even function, it follows that

$$h(x) \le h(0) \qquad (x \in T \cap \mathbf{R}). \tag{15}$$

Combining (13), (14) and (15), we have proved that

$$u(z) \le |\operatorname{Im} z| + h(0) \qquad (z \in T),$$

and all that remains is to evaluate the constant $h(0)$. This we do using (12):

$$\begin{aligned} h(0) &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{|t|}{\cosh(\pi t/2)}\, dt = 2\int_0^{\infty} te^{-\pi t/2}(1+e^{-\pi t})^{-1}\, dt \\ &= 2\sum_{k=0}^{\infty}(-1)^k \int_0^{\infty} te^{-(k+\frac{1}{2})\pi t}\, dt = \frac{8}{\pi^2}\sum_{k=0}^{\infty}(-1)^k(2k+1)^{-2}. \end{aligned}$$

This completes the proof of the lemma. ∎

4. Completion of proofs

In this section we combine the elementary inequalities of Section 2 with the function theory of Section 3 to complete the proofs of the three theorems in Section 1. We shall exploit repeatedly the fact that if $z \mapsto a^z\colon H \to A$ is holomorphic then $z \mapsto \log \|a^z\|\colon H \to [-\infty, \infty)$ is subharmonic (see e.g. [R, Lemma 6.4.1]).

Proof of Theorem 1.1: Fix $w \in H$, and define $h\colon H \to [0, \infty]$ by

$$h(x+iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x}{x^2+(y-t)^2} \log^+ \|a^{w+1+it}\|\,dt.$$

Lemma 3.1, together with the hypothesis (3), implies that h is a positive harmonic function on H, and that

$$\lim_{z\to\zeta} h(z) = \log^+ \|a^{w+1+\zeta}\| \qquad (\zeta \in \partial H).$$

Define $u\colon H \to [-\infty, \infty)$ by

$$u(z) = \log \|a^{w+1+z}\| - h(z).$$

Then u is subharmonic on H, and

$$\limsup_{z\to\zeta} u(z) \le 0 \qquad (\zeta \in \partial H).$$

Also, by Lemma 2.3,

$$u^+(z) = O(|z|^2) \qquad \text{as } |z| \to \infty,$$

and by Lemma 2.1,

$$\limsup_{r\to\infty} u(re^{i\theta})/r < \infty \qquad (\theta \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2})).$$

Therefore we can apply Lemma 3.2 to deduce that

$$u(z) \le L(\operatorname{Re} z) \qquad (z \in H),$$

where, by Corollary 2.2,

$$L = \limsup_{r\to\infty} u(r)/r \le \log \rho.$$

Hence, we conclude that for $z, w \in H$, $z = x + iy$,

$$\log \|a^{w+1+z}\| \leq (\log \rho)x + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x}{x^2+(y-t)^2} \log^+ \|a^{w+1+it}\|\, dt. \tag{16}$$

Now putting $z = 1$ in the above inequality, we obtain

$$\begin{aligned} \log \|a^{w+2}\| &\leq \log \rho + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log^+ \|a^{w+1+it}\|}{1+t^2}\, dt \\ &\leq \log \rho + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log^+ \|a^{1+it}\|}{1+t^2}\, dt + \log^+ \|a^w\|, \end{aligned}$$

which, after taking exponentials, gives

$$\|a^{w+2}\| \leq \rho\sigma \max(\|a^w\|, 1) \qquad (w \in H).$$

Repeating the argument with the semigroup $(a^z)_{z \in H}$ replaced by $(\lambda^z a^z)_{z \in H}$, where $\lambda > 1$, we obtain

$$\|a^{w+2}\| \leq \rho\sigma \max(\|a^w\|, \lambda^{-\operatorname{Re} w}) \qquad (w \in H).$$

Finally, letting $\lambda \to \infty$ gives (4) and completes the proof. ∎

Proof of Theorem 1.3: Suppose first that $\operatorname{Re} z$ is an even integer, say $z = 2n+iy$, where $n \geq 1$ and $y \in \mathbf{R}$. Applying Theorem 1.1 $(n-1)$ times and then Lemma 2.4, we have

$$\|a^{2n+iy}\| \leq (\rho\sigma)^{n-1}\sigma^{2+\frac{\pi}{2}|y|}, \tag{17}$$

which is just the inequality (7) for this z with $K = 1$.

For the remaining z we proceed as follows. Let $k \geq 1$ be an integer, and let T_k be the strip

$$T_k = \{z \in \mathbf{C}\colon 2k < \operatorname{Re} z < 2k+2\}.$$

Define $u\colon T_k \to [-\infty, \infty)$ by

$$u(z) = \log \|a^z\| - (\log \rho)(\operatorname{Re} z/2 - 1) - (\log \sigma)(\operatorname{Re} z/2 + 1).$$

Then u is subharmonic on T_k, and taking $n = k, k+1$ in the inequality (17) shows that

$$\limsup_{z \to \zeta} u(z) \leq \left(\frac{\pi}{2}\log \sigma\right)|\operatorname{Im} \zeta| \qquad (\zeta \in \partial T_k).$$

Also, we certainly have $u^+(z) = O(e^{\alpha|z|})$ for some $\alpha < \pi/2$, since from Lemma 2.3 we even know that $u^+(z) = O(|z|^2)$. Therefore we may apply Lemma 3.3 to a multiple of u to deduce that

$$u(z) \le \Big(\frac{\pi}{2}\log\sigma\Big)\big(|\operatorname{Im} z| + 8\kappa/\pi^2\big) \qquad (z \in T_k),$$

which, when unravelled, gives the inequality (7) for $z \in T_k$ with $K = 1 + 4\kappa/\pi$. As the strips $(T_k)_{k\ge1}$ cover the region $\{z\colon \operatorname{Re} z \ge 2,\ \operatorname{Re} z \text{ not an even integer}\}$, this completes the proof. ∎

Proof of Theorem 1.5: This is the same as the proof of Theorem 1.1 up as far as the inequality (16), but now we treat it in a different way. Using Lemma 3.1 (iii), we have, for all $z, w \in H$,

$$\log\|a^{w+1+z}\| \le (\log\rho)(\operatorname{Re} z) + \Big(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\log^+\|a^{w+1+it}\|}{1+t^2}\,dt\Big)\frac{\big(|z-1|+|z+1|\big)^2}{4\operatorname{Re} z}.$$

We would like to let $w \to 0$ in this inequality, but it is not clear whether we are justified in taking the limit inside the integral. However, under the extra hypothesis (8), we know that there is a sequence (w_n) tending to 0 in H such that $\|a^{w_n}\|$ remains bounded. Thus if we let $w \to 0$ through this particular sequence, then the dominated convergence theorem is applicable, and we may indeed conclude that

$$\log\|a^{1+z}\| \le (\log\rho)(\operatorname{Re} z) + (\log\sigma)\big(|z-1|+|z+1|\big)^2/(4\operatorname{Re} z) \qquad (z \in H).$$

The inequality (9) now follows on replacing z by $z-1$ throughout and taking exponentials. ∎

5. Concluding remarks and questions

(i) To give a direct proof of the original results of Esterle and Sinclair, all that is needed is to follow the proof of Theorem 1.1 as far as the inequality (16), and then use the fact that in a radical Banach algebra $\log\rho = -\infty$. Such a proof was already pointed out in [R, Theorem 6.4.9]. Note that the place of the Ahlfors–Heins theorem from the original proofs has now been taken by the more elementary Lemma 3.2.

(ii) Dales and Hayman [DH] have refined Esterle's technique to prove the tauberian theorem for Beurling algebras. Is there also a quantitative version of their result?

(iii) To what extent, if any, are the inequalities (4), (7) and (9) sharp? In particular, are the exponents of ρ and σ in (7) anywhere near the best possible?

(iv) Given ρ, σ as in Theorem 1.1, is it possible to find bounds for $\|a^z\|$ when $\operatorname{Re} z < 1$? The answer is no, because of the following simple example shown to the author by Michael White. Let $\gamma > 0$, let $\|\cdot\|$ be the norm on $\mathbf{C}$ given by $\|w\| = e^\gamma |w|$, and let $(a^z)_{z\in H}$ be the holomorphic semigroup in $\mathbf{C}$ given by $a^z = e^{-\gamma z}$. Then $\|a^{1+iy}\| = 1$ for all $y \in \mathbf{R}$, so $\sigma = 1$ and $\rho \leq 1$ (in fact $\rho = e^{-\gamma}$). However, given z with $0 < \operatorname{Re} z < 1$, we have $\|a^z\| = e^{\gamma(1-\operatorname{Re} z)}$, which can be arbitrarily large depending on the choice of γ. (Note that although the norm is submultiplicative, it is not quite an algebra-norm on $\mathbf{C}$ because $\|1\| \neq 1$, but we can circumvent this problem simply by adjoining to $\mathbf{C}$ a new identity of norm 1.)

(v) Here is a related question. If (3) holds, then it is clear that for $x > 1$

$$\int_{-\infty}^{\infty} \frac{\log^+ \|a^{x+iy}\|}{1+y^2}\, dy < \infty,$$

but can we also conclude that the integral is finite if $x < 1$?

(vi) Suppose that $(a^x)_{x\geq 0}$ is a C_0-semigroup of bounded linear operators on a complex Banach space. Then one can associate to it its infinitesimal generator, from which, in turn, the original semigroup can be recaptured (see e.g. [HP, Chapters 10–12]). Thus, in principle, *all* the properties of the semigroup are reflected in the infinitesimal generator. For example, the property of having a (certain type of) holomorphic extension $(a^z)_{z\in H}$ to H can be characterized in terms of the infinitesimal generator using a version of the Hille–Yosida theorem (see e.g. [HP, Theorem 12.8.1]). It would be interesting to find an analogous characterization of the property (3).

(vii) An inspection of the proofs in this paper shows that we have exploited the semigroup property (1) only through its weaker consequence that

$$\|a^{z+w}\| \leq \|a^z\|\|a^w\| \qquad (z, w \in H),$$

and also we have used the holomorphicity of $z \mapsto a^z$ only via the subharmonicity (and continuity) of $z \mapsto \log \|a^z\|$. Thus, the whole paper could in fact have been written purely in terms of subadditive subharmonic functions. This suggests perhaps that, to answer some of the questions posed above, we should apply the definition of holomorphic semigroup more directly than has been tried so far.

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